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The Tzitzeica equation: A Bäcklund transformation interpreted as truncated Painlevé expansion

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Abstract. It is shown that the overdetermined system of equations obtained by truncating the Painlevé expansion for the classical Tzitzeica equation admits a solution which defines a Bäcklund transformation.

The partial differential equation

$$\omega_{xy} = \mathrm{e}^{\omega} - \mathrm{e}^{-2\omega} \tag{1}$$

was derived in a classical geometric context as long ago as 1910 by Tzitzeica [1]. Indeed, even a linear representation and a Bäcklund transformation are cited therein. Seventy years later, the Tzitzeica equation was rediscovered and subsequently analysed by several authors [2–5] in the setting of modern soliton theory. In terms of other physical applications, Gaffet [6] has shown that, for a particular class of gas laws, a $(1+1)$-dimensional anisentropic gas dynamics system may be reduced to the Tzitzeica equation.

The Painlevé test (and its modifications) for partial differential equations as introduced by Weiss *et al* [7] has become widely accepted as a test for integrability [8]. In fact, in 1986, Weiss [9] proved that the Tzitzeica equation passes the Painlevé test. However, the choice therein of the new dependent variable $\mathrm{e}^{-\omega}$ in order to rationalize equation (1) led to the erroneous conclusion that the Tzitzeica equation does not possess a Bäcklund transformation. Later, Musette and Conte [10] pointed out that the Tzitzeica equation in the form

$$hh_{xy} - h_x h_y = h^3 - 1 \qquad h = \mathrm{e}^{\omega} \tag{2}$$

does indeed admit a non-trivial truncated Painlevé expansion of the type commonly associated with a Bäcklund transformation, namely

$$h = 2\frac{\phi_x\phi_y}{\phi^2} - 2\frac{\phi_{xy}}{\phi} + H. \tag{3}$$

However, up to now, the consistency of the overdetermined system of equations for the singularity manifold function ϕ has not been established. This system, which is obtained by inserting the ansatz (3) into (2) and equating the coefficients of the various powers of ϕ with zero, reads

$$\phi^0: \qquad HH_{xy} - H_x H_y = H^3 - 1 \tag{4a}$$

$$\phi^{-1}: \qquad H\phi_{xxyy} - H_y\phi_{xxy} - H_x\phi_{xyy} + H_{xy}\phi_{xy} - 3H^2\phi_{xy} = 0 \tag{4b}$$

$$\phi^{-2}: \qquad E_2[H,\phi] = 0 \tag{4c}$$

$$\phi^{-3}: \qquad E_3[H,\phi] = 0. \tag{4d}$$

Here, E_2 and E_3 indicate involved functions of H and derivatives of H and ϕ. Consistency, in turn, implies that any solution of the system (4) defines a Bäcklund transformation between two solutions H and h of the Tzitzeica equation (2) via

$$h = H - 2(\ln\phi)_{xy}. \tag{5}$$

In this paper, we give *a* solution to the overdetermined system (4). The significance of this result becomes apparent in view of the conjecture that Lax pairs and Bäcklund transformations of integrable differential equations may be obtained via truncation (at constant level) of Painlevé expansions [9]. The result is as follows.

Theorem. Let H be a solution of the Tzitzeica equation (4a) and φ, ψ be solutions of the corresponding mutually 'adjoint' linear triads

$$\varphi_{xx} = \frac{H_x}{H}\varphi_x + \frac{\lambda}{H}\varphi_y \qquad \psi_{xx} = \frac{H_x}{H}\psi_x - \frac{\lambda}{H}\psi_y \tag{6a}$$

$$\varphi_{xy} = H\varphi \qquad \psi_{xy} = H\psi \tag{6b}$$

$$\varphi_{yy} = \frac{H_y}{H}\varphi_y + \frac{\lambda^{-1}}{H}\varphi_x \qquad \psi_{yy} = \frac{H_y}{H}\psi_y - \frac{\lambda^{-1}}{H}\psi_x \tag{6c}$$

respectively. Then, ϕ defined by

$$\phi_x = \psi_x\varphi - \psi\varphi_x \qquad \phi_y = \psi\varphi_y - \psi_y\varphi \tag{7}$$

satisfies the remaining equations (4b)–(4d) provided that φ and ψ obey the constraint

$$\psi_y\varphi_x + \psi_x\varphi_y - H\psi\varphi = 0. \tag{8}$$

Proof. It is noted that the systems (6) are compatible if, and only if, H is a solution of the Tzitzeica equation. On the other hand, the quantity ϕ is well-defined since the defining equations (7) are compatible modulo the triads (6). Finally, the constraint (8) is admissible as the left-hand side of (8) divided by H is a first integral of (6). It is readily verified (using MAPLE) that ϕ as given by (7) is indeed a solution of (4b)–(4d).

The nature of the Bäcklund transformation defined by equations (5)–(8) is as follows. The Bäcklund transformation given by Tzitzeica in [1] is based on the classical Moutard transformation for hyperbolic equations of the form (6b) and reads

$$H' = H - 2(\ln\varphi)_{xy}. \tag{9}$$

The Moutard transformation may be iterated in a purely algebraic manner in terms of skew-symmetric potentials of the form adopted by ϕ in (7). Indeed, the Bäcklund transformation presented here turns out to be a particular case of the double Moutard transformation [11, 12]. Against this background, it is emphasized that the fact that equations (5)–(8) constitute a Bäcklund transformation for the Tzitzeica equation and that ϕ is defined only up to an additive constant of integration guarantees that the conditions $E_i = 0$ are satisfied.

That equations (6)–(8) constitute the general solution of the overdetermined system (4) is yet to be established.

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